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On H_0^1 -estimates for radially symmetric solutions
of semilinear elliptic equations

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1. Introduction. We consider the radially symmetric solutions for the semilinear elliptic equation

$$(1.1) \quad \Delta u + g(u) = 0, \quad x \in \Omega,$$

$$(1.2) \quad u = 0, \quad x \in \partial\Omega,$$

where $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$, $n \geq 2$, and $g(s)$ is a continuous function such that

$$(g_1) \quad g(0) = 0, \quad \lim_{|s| \rightarrow \infty} g(s)/s = \infty \quad \text{and}$$

$$g'(0) = \lim_{s \rightarrow 0} g(s)/s \quad \text{exists.}$$

The equation for radially symmetric solutions $u = u(t)$, $t = |x|$, is the following form.

$$(1.3) \quad u'' + \frac{n-1}{t} u' + g(u) = 0, \quad t \in (0, 1),$$

$$(1.4) \quad u'(0) = 0, \quad u(1) = 0.$$

Under some conditions on $g(s)$, Ambrosetti and Rabinowitz [1] established that for any bounded domain Ω the problem (1.1)-(1.2) possesses infinitely many solutions and moreover $H_0^1(\Omega)$ norms of solutions assume arbitrarily large values. The related problems are treated in [2, 5, 8, 10, 12, 13, 14]. In

the case where Ω is the unit ball, the existence of infinitely many radially symmetric solutions has been investigated by Castro-Kurepa [4] and Struwe [15] (see [3] for $n = 1$). In fact, Struwe [15] has proved by a variational method that there is an integer k_0 such that for any $k \geq k_0$ the problem (1.3)-(1.4) admits a solution with exactly k zeros in $[0, 1]$. On the other hand, using a shooting method, Castro and Kurepa [4] have showed the same results under weaker assumptions on $g(s)$. Therefore we will study the relation between $H_0^1(\Omega)$ norms of radially symmetric solutions and the numbers of their zeros.

2. Main results. Recall that Ω is the unit ball in \mathbb{R}^n . We denote by $L^r(\Omega)$ ($1 \leq r \leq \infty$) and by $H_0^1(\Omega)$ the usual Lebesgue and Sobolev spaces, respectively. The norm of $L^r(\Omega)$ is denoted by $\|\cdot\|_r$. The $H_0^1(\Omega)$ norm is defined by

$$\|u\|_{H_0^1(\Omega)} = \|\nabla u\|_2 = \left(\int_{\Omega} |\nabla u(x)|^2 dx \right)^{1/2}.$$

Let H be the subspace of $H_0^1(\Omega)$ which consists of radially symmetric functions. We define the norm $\|\cdot\|_H$ of H by

$$\|u\|_H = \omega_n^{-1/2} \|u\|_{H_0^1(\Omega)} = \left(\int_0^1 u'(t)^2 t^{n-1} dt \right)^{1/2},$$

where ω_n means the surface area of the unit sphere $\partial\Omega$. We write S for the set of all solutions $u \in C^2(0, 1) \cap C^1[0, 1]$ of (1.3)-(1.4). For $k \in \mathbb{N}$, S_k denotes the set of all solutions which have exactly k zeros in the interval $[0, 1]$. Let n^* be defined by $n^* = \infty$ if $n = 2$ and $n^* = \frac{n+2}{n-2}$ if $n \geq 3$. We state our first result which gives the lower bounds of solutions.

Theorem 1 (lower estimates) Suppose the condition (g_1) and there are constants $p \in (1, n^*)$ and $a_i > 0$ such that

$$(g_2) \quad sg(s) \leq a_1 |s|^{p+1} + a_2 \quad \text{for all } s \in \mathbb{R}.$$

Then there are constants $c_1, c_2 > 0$ such that

$$c_1 k^{(p+1)/(p-1)} - c_2 \leq \|u\|_H \quad \text{for any } u \in S_k \text{ and } k \geq 1.$$

Theorem 1 may assert nothing for small k in the case of $c_1 < c_2$. However, as stated below, we see that any nontrivial solution is bounded away from the trivial solution.

Theorem 2. In addition to conditions (g_1) and (g_2) suppose that $g'(0)$ is not an eigenvalue of the problem:

$$(2.1) \quad -v'' - \frac{n-1}{t} v' = \lambda v, \quad t \in (0, 1),$$

$$(2.2) \quad v'(0) = v(1) = 0.$$

Then there is a constant $c > 0$ such that

$$c \leq \|u\|_H \quad \text{for all } u \in S \setminus \{0\}.$$

The result stated in Theorem 1 is optimal, since we also have upper estimates for the solutions.

Theorem 3 (upper estimates). Set $G(s) = \int_0^s g(r) dr$. In addition to (g_1) suppose the following two conditions:

$$(g_3) \quad \limsup_{|s| \rightarrow \infty} \frac{sg(s)}{G(s)} < n^* + 1.$$

There are constants $p \in (1, n^*)$, $b_i > 0$ and $R > 0$ such that

$$(g_4) \quad b_1 |s|^{p+1} \leq sg(s) \leq b_2 |s|^{p+1} \quad \text{for all } |s| \geq R.$$

Then there is a constant $c > 0$ such that

$$\|u\|_H \leq ck^{(p+1)/(p-1)} \quad \text{for any } u \in S_k \text{ and } k \geq 1.$$

Finally, we consider the Emden-Fowler equation which involves a typical nonlinear term $g(s)$.

Example (Emden-Fowler Equation). Consider the equation

$$(2.3) \quad u'' + \frac{n-1}{t} u' + |u|^{p-1} u = 0, \quad t \in (0, 1),$$

$$(2.4) \quad u'(0) = u(1) = 0,$$

where $p \in (1, n^*)$. We shall show that for each $k \geq 1$ the problem (2.3)-(2.4) possesses a unique solution which has exactly k zeros in $[0, 1]$ and satisfies $u(0) > 0$. If we denote the solution by $u_k(t)$, then it follows that

$$S_k = \{u_k, -u_k\} \quad \text{and} \quad S = \{0\} \cup \{\pm u_k : k \in \mathbb{N}\}.$$

To prove these assertions, we consider (2.3) on $[0, \infty)$ together with the initial condition,

$$(2.5) \quad u'(0) = 0 \quad \text{and} \quad u(0) = 1.$$

It is easy to verify that equation (2.3) with (2.5) possesses a unique global solution $w(t)$ on $[0, \infty)$. Furthermore the solution $w(t)$ is oscillatory (see [11, Corollary 6.7]). Here $w(t)$ is said to be oscillatory if $w(t)$ has an unbounded sequence of zeros in $[0, \infty)$. Since $w(t)$ has at most finite zeros in any bounded interval, we may denote the set of all zeros of $w(t)$ as $\{s_k\}_{k=1}^{\infty}$ ($0 < s_1 < s_2 < \dots \uparrow \infty$). First we observe that $\lambda^{2/(p-1)} w(\lambda t)$ ($\lambda > 0$), as well as $w(t)$, satisfies

equation (2.3) on $[0, \infty)$. Set $u_k(t) \equiv s_k^{2/(p-1)} w(s_k t)$. Then the function $u_k(t)$ is a solution of (2.3)-(2.4) which has exactly k zeros in $[0, 1]$ and satisfies $u_k(0) > 0$. Such a solution is unique, since any solution of (2.3) satisfying $u'(0) = 0$ and $u(0) > 0$ can be written in the form $\lambda^{2/(p-1)} w(\lambda t)$ ($\lambda > 0$). In view of these facts we see that $S_k = \{u_k, -u_k\}$ and $S = \{0\} \cup \{\pm u_k : k \in \mathbb{N}\}$. We now apply Theorems 1 through 3 to find the asymptotic distribution in $H_0^1(\Omega)$ of solutions of (2.3)-(2.4). That is, there exist constants $c_1, c_2 > 0$ such that

$$c_1 k^{(p+1)/(p-1)} \leq \|u_k\|_H \leq c_2 k^{(p+1)/(p-1)} \quad \text{for any } k \in \mathbb{N}.$$

3. Proofs of Theorems 1 and 2

Proof of Theorem 2. We prove Theorem 2 by contradiction. Suppose that there is a sequence of nontrivial solutions $\{u_j\}_{j=1}^\infty \subset S \setminus \{0\}$ such that $\lim_{j \rightarrow \infty} \|u_j\|_H = 0$. Then using "Moser's iteration technique" (cf. [6, 9]), we have $\lim_{j \rightarrow \infty} \|u_j\|_\infty = 0$. Set $v_j(x) = \|u_j\|_\infty^{-1} u_j(x)$; each v_j satisfies

$$-\Delta v_j = (g(u_j)/u_j) v_j \quad \text{for } x \in \Omega.$$

Since the right hand side is bounded in $L^\infty(\Omega)$, $\{v_j\}_{j=1}^\infty$ is bounded in $W^{2,q}(\Omega)$ and is relatively compact in $W^{1,q}(\Omega)$ for any $q \in [1, \infty)$. Hence one finds a subsequence of $\{v_j\}_{j=1}^\infty$ which converges to some function $v(x)$ in both $W^{1,q}(\Omega)$ and $L^\infty(\Omega)$. This together with $\|v_j\|_\infty = 1$ implies

$$-\Delta v = g'(0)v, \quad (x \in \Omega); \quad v = 0, \quad (x \in \partial\Omega); \quad \|v\|_\infty = 1.$$

Since $v(x)$ is radially symmetric, $g'(0)$ becomes an eigenvalue

of (2.1)-(2.2). This contradicts the assumptions of Theorem 2 and the proof is complete. \square

To prove Theorem 1 we need the following two lemmas.

Lemma 1. *For each $u \in S$, we have*

$$(3.1) \quad \int_0^1 u'(t)^2 t^{n-1} dt = \int_0^1 u(t)g(u(t))t^{n-1} dt.$$

Proof. Multiplying (1.3) by $u(t)t^{n-1}$ and applying integration by parts, we obtain the desired relation (3.1). \square

The next lemma is obtained by applying Sturm's comparison theorem.

Lemma 2 ([7, p346, Corollary 5.2]). *Let $q(t)$ be a continuous function on $[a, b]$. Let $v(t) \not\equiv 0$ be a solution of the equation:*

$$v'' + q(t)v = 0, \quad t \in [a, b].$$

Assume that $v(t)$ has exactly k zeros in $(a, b]$. Then

$$k < \frac{1}{2} \left((b-a) \int_a^b q^+(t) dt \right)^{1/2} + 1,$$

where $q^+(t) \equiv \max\{q(t), 0\}$.

Proof of Theorem 1. We shall prove Theorem 1 in the simple case only where $g(s) = |s|^{p-1}s$, since the same methods are valid for the general case of $g(s)$ as well (see [9]). That is, we consider the Emden-Fowler equation (2.3)-(2.4). Let $u \in S_k$. In what follows, we denote various constants independent of u and k by $C (> 0)$. In the case of $n \geq 3$, we employ

the following Liouville transformation:

$$r = t^{1/\alpha}, \quad v(r) = r^\beta u(t), \quad 2\beta = (n-2)\alpha + 1,$$

where $\alpha (> 1)$ is a constant to be determined later. Then (2.3) is reduced to

$$v''(r) + q(r)v(r) = 0, \quad r \in (0, 1],$$

where $q(r) \equiv \alpha^2 r^{2\alpha-2} |u(r^\alpha)|^{p-1} - \beta(\beta-1)r^{-2}.$

Since $q(r)$ and $v(r)$ satisfy the assumptions of Lemma 2 on the interval $[\varepsilon, 1]$ for sufficiently small $\varepsilon > 0$, it follows that

$$\begin{aligned} k &< \frac{1}{2} \left\{ (1-\varepsilon) \int_{\varepsilon}^1 q^+(r) dr \right\}^{1/2} + 1 \leq \frac{1}{2} \left\{ \int_0^1 q^+(r) dr \right\}^{1/2} + 1 \\ &\leq C \left\{ \int_0^1 |u(r^\alpha)|^{p-1} r^{2\alpha-2} dr \right\}^{1/2} + 1. \end{aligned}$$

From Hölder's inequality,

$$\begin{aligned} (3.2) \quad k &\leq C \left\{ \int_0^1 |u(t)|^{p+1} t^{n-1} dt \right\}^{(p-1)/2(p+1)} \\ &\quad \times \left\{ \int_0^1 t^{\gamma(p+1)/2} dt \right\}^{1/(p+1)} + C, \end{aligned}$$

where $\gamma \equiv 1 - 1/\alpha - (n-1)(p-1)/(p+1)$. Since $p \in (1, n^*)$, one can choose $\alpha > 1$ so large that $\gamma(p+1)/2 > -1$. Then the function $t^{\gamma(p+1)/2}$ is integrable over $[0, 1]$. From (3.2) and (3.1) we obtain the desired estimate for $n \geq 3$. In the case of $n = 2$, we employ the Liouville transformation

$$r = \frac{1}{1 - \log t}, \quad v(r) = ru(t),$$

to reduce (2.3) to the equation,

$$v'' + q(r)v = 0, \quad r \in (0, 1],$$

where $q(r) \equiv t^2 |\log(t/e)|^4 |u(t)|^{p-1}$ with $t = e^{(r-1)/r}$.

Applying Lemma 2 and using Hölder's inequality, we obtain the assertion for $n = 2$ in the same way as in the case of $n \geq 3$. The proof is complete. \square

4. Proof of Theorem 3 The purpose of this section is to prove Theorem 3. Here we deal with the only case where $g(s) = |s|^{p-1}s$. However, our methods are applicable to the nonlinear term $g(s)$ satisfying the assumptions of Theorem 3 (see [9]). In order to prove Theorem 3 we need several lemmas.

Lemma 3. *There are constants $\theta > 0$ and $c_i > 0$ ($1 \leq i \leq 3$) such that*

$$\begin{aligned} \max_{0 \leq t \leq 1} |u(t)|^{p+1} t^n &\leq c_1 \int_0^1 |u(t)|^{p+1} t^{n-1} dt \\ &\leq c_2 \int_0^1 |u(t)|^{p+1} t^{2n-3+\theta} dt \leq c_3 \max_{0 \leq t \leq 1} |u(t)|^{p+1} t^n \end{aligned}$$

for any $u \in S$.

Proof. Multiplying (2.3) by $u'(t)t^m$ ($m > 0$) and integrating over $[0, T]$, we have

$$\begin{aligned} (4.1) \quad &\frac{1}{2} u'(T)^2 T^m + \frac{1}{p+1} |u(T)|^{p+1} T^m \\ &= \frac{m}{p+1} \int_0^T |u(t)|^{p+1} t^{m-1} dt + \left(\frac{m}{2} + 1 - n\right) \int_0^T u'(t)^2 t^{m-1} dt. \end{aligned}$$

First we substitute $m = n$ into (4.1) to obtain

$$|u(T)|^{p+1} T^n \leq n \int_0^T |u(t)|^{p+1} t^{n-1} dt \leq n \int_0^1 |u(t)|^{p+1} t^{n-1} dt$$

for all $T \in [0, 1]$, which implies the first inequality of Lemma

3. Secondly, the substitution of $m = n$ and $T = 1$ into (4.1) yields

$$(4.2) \quad \frac{1}{2} u'(1)^2 = \left(\frac{n}{p+1} - \frac{n-2}{2} \right) \int_0^1 |u(t)|^{p+1} t^{n-1} dt,$$

where we have used (3.1). Thirdly, substituting $m = 2n-2+\theta$ and $T = 1$ into (4.1), we have

$$(4.3) \quad \begin{aligned} \frac{1}{2} u'(1)^2 &= \frac{2n-2+\theta}{p+1} \int_0^1 |u(t)|^{p+1} t^{2n-3+\theta} dt \\ &\quad + \frac{\theta}{2} \int_0^1 u'(t)^2 t^{2n-3+\theta} dt \\ &\leq \frac{2n-2+\theta}{p+1} \int_0^1 |u(t)|^{p+1} t^{2n-3+\theta} dt \\ &\quad + \frac{\theta}{2} \int_0^1 |u(t)|^{p+1} t^{n-1} dt, \end{aligned}$$

where we have used (3.1) and $n-2+\theta > 0$. Combining (4.2) and (4.3), one finds

$$(4.4) \quad \begin{aligned} &\left(\frac{n}{p+1} - \frac{n-2+\theta}{2} \right) \int_0^1 |u(t)|^{p+1} t^{n-1} dt \\ &\leq \frac{2n-2+\theta}{p+1} \int_0^1 |u(t)|^{p+1} t^{2n-3+\theta} dt. \end{aligned}$$

Since $p \in (1, n^*)$, we can choose $\theta > 0$ so small that $n/(p+1) - (n-2+\theta)/2 > 0$. Then (4.4) implies the second inequality of Lemma 3. Finally it is not difficult to check the last inequality of Lemma 3, and the proof is complete. \square

In view of Lemma 3, we introduce the next notation for convenience.

Definition 1. We define

$$M(u) \equiv \max_{0 \leq t \leq 1} |u(t)|^{p+1} t^n \quad \text{for } u \in S.$$

By Lemma 3 and (3.1) there is a constant $c_4 > 0$ such that

$$(4.5) \quad \int_0^1 u'(t)^2 t^{n-1} dt \leq c_4 M(u)$$

for any $u \in S$. Hence, to prove Theorem 3, it is sufficient to compute the value of $M(u)$ instead of the $H_0^1(\Omega)$ norm of u .

The next lemma, which follows readily from Sturm's comparison theorem, is useful for proving Lemma 5 below.

Lemma 4 (cf. [7, p336, Exercise 3.2]). *In the differential equation*

$$(4.6) \quad (p(t)u')' + q(t)u = 0, \quad t \in [a, b],$$

let $p(t) \in C^1[a, b]$ and $q(t) \in C[a, b]$ satisfy

$$p(t) \geq m \quad \text{and} \quad M \geq q(t),$$

where m and M are positive constants. If $u(t)$ is a solution with a pair of zeros t_1, t_2 ($t_1 < t_2$) and $u(t) \neq 0$, then we have

$$t_2 - t_1 \geq \pi \left(\frac{m}{M} \right)^{1/2}.$$

To evaluate $M(u)$ we subdivide the interval $[0, 1]$ in the following way:

Definition 2. Let $u \in S_k$ and $\{t_i\}_{i=1}^k$ ($0 < t_1 < t_2 < \dots < t_k = 1$) denote its zeros. Let $\delta \in (0, 1/2)$. Moreover we set

$$\delta_i = \delta(t_{i+1} - t_i), \quad \delta_0 = 2\delta t_1,$$

$$I_i = [t_i + \delta_i, t_{i+1} - \delta_i], \quad 1 \leq i \leq k-1,$$

$$I_0 = [0, t_1 - \delta_0],$$

$$I = \bigcup_{i=0}^{k-1} I_i \quad \text{and} \quad J = [0, 1] \setminus I.$$

In what follows, we denote various constants by C_ε , C_δ , $C_{\varepsilon, \delta}$ and C ; C_ε means a constant depending upon ε , $C_{\varepsilon, \delta}$ represents a constant depending possibly on ε and δ , and C denotes an absolute constant.

Lemma 5. *Let $u \in S_k$ and $\varepsilon \in (0, 1)$. Then we have*

$$(i) \quad \int_{I_i} |u(t)|^{p-1} t^{n-1} dt \leq \frac{C_\delta}{t_{i+1} - t_i}, \quad 1 \leq i \leq k-1;$$

$$(ii) \quad \int_{I_0} |u(t)|^{p-1} t^{n-1} dt \leq C_\delta;$$

$$(iii) \quad \text{if } t_i \geq \varepsilon, \text{ then } \frac{1}{t_{i+1} - t_i} \leq C_\varepsilon M(u)^{(p-1)/2(p+1)}.$$

Proof. Consider the eigenvalue problem

$$-(t^{n-1}v')' - a(t)v = \lambda t^{n-1}v, \quad t \in (\alpha, \beta),$$

$$v(\alpha) = v(\beta) = 0,$$

where $\alpha = t_i$, $\beta = t_{i+1}$, $i \geq 1$ and $a(t) \equiv |u(t)|^{p-1} t^{n-1}$.

We see that $\lambda = 0$ is the first eigenvalue. In fact, $v(t) = u(t)$ and $\lambda = 0$ satisfy the above equation and moreover $u(t)$ has no zeros in the interval $(\alpha, \beta) = (t_i, t_{i+1})$. Therefore $\lambda = 0$ is the first eigenvalue, and so we have the following variational characterization:

$$(4.7) \quad \min_{v \in V} \frac{\int_{\alpha}^{\beta} \{v'(t)^2 t^{n-1} - a(t)v(t)^2\} dt}{\int_{\alpha}^{\beta} v(t)^2 t^{n-1} dt} = 0,$$

where V denotes the set of all functions $v \in C^1[\alpha, \beta]$ satisfying $v(\alpha) = v(\beta) = 0$ and $v \not\equiv 0$. It follows from (4.7) that

$$\begin{aligned} \int_{\alpha}^{\beta} v'(t)^2 t^{n-1} dt &\geq \int_{\alpha}^{\beta} a(t) v(t)^2 dt \\ &\geq \left(\min_{t \in I_i} v(t)^2 \right) \int_{I_i} a(t) dt \end{aligned}$$

for any $v \in V$. We choose $v(t) \equiv (\beta - t)(t - \alpha) \in V$ to get

$$\frac{1}{3}(\beta - \alpha)^3 \geq \left(\frac{\delta^2}{4} \right) (\beta - \alpha)^4 \int_{I_i} a(t) dt.$$

This implies the assertion (i). Next we prove the assertion (ii). In the same argument as above we have

$$\int_0^{t_1} v'(t)^2 t^{n-1} dt \geq \left(\min_{t \in I_0} v(t)^2 \right) \int_{I_0} a(t) dt$$

for any $v \in C^1[0, 1]$ satisfying $v'(0) = v(t_1) = 0$.

Substituting $v(t) \equiv t_1^2 - t^2$, we obtain the assertion (ii).

We then prove the last assertion (iii). The function $u(t) \in S_k$ satisfies the differential equation (4.6) with $p(t) = t^{n-1}$ and $q(t) = |u(t)|^{p-1} t^{n-1}$. The functions $p(t)$ and $q(t)$ are estimated as

$$p(t) \geq \varepsilon^{n-1} \quad \text{and} \quad q(t) \leq C_{\varepsilon} M(u)^{(p-1)/(p+1)} \quad t \in [t_i, t_{i+1}],$$

where we used $t_i \geq \varepsilon$. Therefore assertion (iii) follows from Lemma 4. The proof is thereby complete. \square

Lemma 6. Let $\theta (> 0)$ be defined as in Lemma 3. Let $\varepsilon \in (0, 1)$. Then for $t_j \geq \varepsilon$, we get

$$\sum_{i=j}^{k-1} \int_{I_i} |u(t)|^{p+1} t^{2n-3+\theta} dt \leq C_{\varepsilon, \delta} k M(u)^{(p+3)/2(p+1)}.$$

Proof. Let $t_i \geq \varepsilon$. We set $\xi = \theta - 2 + n(p-1)/(p+1)$.

Then Lemma 5 implies

$$\begin{aligned} & \int_{I_i} |u(t)|^{p+1} t^{2n-3+\theta} dt \\ &= \int_{I_i} \left(|u(t)|^{p+1} t^n \right)^{2/(p+1)} t^\xi |u(t)|^{p-1} t^{n-1} dt \\ &\leq \max(\varepsilon^\xi, 1) M(u)^{2/(p+1)} C_{\varepsilon, \delta} M(u)^{(p-1)/2(p+1)} \\ &\leq C_{\varepsilon, \delta} M(u)^{(p+3)/2(p+1)}. \end{aligned}$$

Summing up both sides with respect to $i = j, j+1, \dots, k-1$, we obtain the desired inequality. This completes the proof. \square

Lemma 7.

$$\begin{aligned} & \int_{[\varepsilon, 1] \cap I} |u(t)|^{p+1} t^{2n-3+\theta} dt \\ &\leq C \varepsilon^a M(u) + C_{\varepsilon, \delta} k^{2(p+1)/(p-1)} \end{aligned}$$

for any $\varepsilon \in (0, 1/2)$, where $a = n - 2 + \theta (> 0)$.

Proof. There are the two cases to be considered.

- (A) There is an integer $i \in [1, k-1]$ such that $t_i \in [\varepsilon, 2\varepsilon]$;
- (B) Any t_i ($1 \leq i \leq k-1$) does not belong to $[\varepsilon, 2\varepsilon]$.

We can find an integer $j \in [1, k-1]$ such that $t_{j-1} < \varepsilon \leq t_j \leq 2\varepsilon$ in case (A), where we understand that $t_0 = 0$. On the

other hand, there is an integer $j \in [1, k]$ such that $t_{j-1} < \varepsilon < 2\varepsilon < t_j$ in case (B). In either case, it follows from Lemma 6 that

$$(4.8) \quad \int_{[\varepsilon, 1] \cap I} |u(t)|^{p+1} t^{2n-3+\theta} dt \\ \leq \int_{[\varepsilon, t_j] \cap I_{j-1}} |u(t)|^{p+1} t^{2n-3+\theta} dt + C_{\varepsilon, \delta} k M(u)^{(p+3)/2(p+1)}.$$

Now suppose $[\varepsilon, t_j] \cap I_{j-1} \neq \emptyset$. We want to estimate

$$K \equiv \int_{[\varepsilon, t_j] \cap I_{j-1}} |u(t)|^{p+1} t^{2n-3+\theta} dt.$$

First, in case (A) it follows from the direct computation that

$$(4.9) \quad K \leq \int_{\varepsilon}^{2\varepsilon} |u(t)|^{p+1} t^{2n-3+\theta} dt \leq \frac{2^a - 1}{a} \varepsilon^a M(u).$$

Secondly, in case (B), we have

$$K \leq C_{\varepsilon} M(u)^{2/(p+1)} K_{j-1},$$

where

$$K_{j-1} \equiv \int_{I_{j-1}} |u(t)|^{p-1} t^{n-1} dt.$$

Since $t_{j-1} < \varepsilon < 2\varepsilon < t_j$, Lemma 5 implies that

$$K_{j-1} \leq \frac{C_{\delta}}{t_j - t_{j-1}} \leq \varepsilon^{-1} C_{\delta} \quad \text{if } j \geq 2, \text{ and} \\ K_{j-1} \leq C_{\delta} \quad \text{if } j = 1.$$

Therefore we obtain

$$(4.10) \quad K \leq C_{\varepsilon, \delta} M(u)^{2/(p+1)}.$$

From (4.8), (4.9) and (4.10), we have

$$\int_{[\varepsilon, 1] \cap I} |u(t)|^{p+1} t^{2n-3+\theta} dt$$

$$\leq C\varepsilon^a M(u) + C_{\varepsilon, \delta} k M(u)^{(p+3)/2(p+1)} + C_{\varepsilon, \delta} M(u)^{2/(p+1)}.$$

Thus, we can apply Young's inequality to the right hand side to get the conclusion. \square

We now give the proof of Theorem 3.

Proof of Theorem 3. We set $a = n - 2 + \theta (> 0)$ as in Lemma 7. Let $\varepsilon, \delta \in (0, 1/2)$. In order to evaluate $M(u)$, we estimate

$$(4.11) \quad \int_0^1 |u(t)|^{p+1} t^{2n-3+\theta} dt = \left(\int_0^\varepsilon + \int_{[\varepsilon, 1] \cap J} + \int_{[\varepsilon, 1] \cap I} \right) |u(t)|^{p+1} t^{2n-3+\theta} dt.$$

First, we see from Definition 2 that the measure of J is 2δ . Therefore the first and the second terms can be estimated as

$$(4.12) \quad \int_0^\varepsilon |u(t)|^{p+1} t^{2n-3+\theta} dt \leq \frac{\varepsilon^a}{a} M(u)$$

and

$$(4.13) \quad \int_{[\varepsilon, 1] \cap J} |u(t)|^{p+1} t^{2n-3+\theta} dt \leq C(\delta^a + \delta) M(u),$$

respectively. The last term on the right hand side of (4.11) has already been estimated in Lemma 7. Using (4.11), (4.12), (4.13) and Lemmas 3 and 7, we have

$$(4.14) \quad M(u) \leq C(\varepsilon^a + \delta^a + \delta) M(u) + C_{\varepsilon, \delta} k^{2(p+1)/(p-1)},$$

where C is independent of u, k, ε and δ . We now choose

$\varepsilon, \delta > 0$ so small that $C(\varepsilon^{\alpha+\delta}\delta^{\alpha+\delta}) < 1/2$. Then (4.14) implies

$$M(u) \leq C k^{2(p+1)/(p-1)}.$$

Thus, the desired assertion is obtained by applying (4.5). The proof is thereby complete. \square

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